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The impedance of scaled transmission lines

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Abstract. It is shown that resistance/capacitance transmission lines with multiplicative scaling relationships in the resistive and capacitive elements exhibit a limited range of spectral responses. At high and low frequencies the characteristic impedance of the first, and in a line of finite length the last, elements of the line dominate. In the intermediate frequency range either constant phase angle response or fractional power response may be obtained. The scaling conditions necessary for the observation of these intermediate-frequency responses are established. It is shown that the scaled transmission lines can be considered as fractal when the scaling derives from fractional dimensional properties in continuous media.

1. Introduction

In recent years the electrical impedances of a number of fractally scaled electrical circuits have been examined, generally as models of specific physical problems. One example is the examination of the impedance of a rough electrode surface immersed in a conducting electrolyte. This work was initiated by Le Mehaute and Crepy (1983) and has developed along a number of parallel courses; the examination of Cantor bar-like scaled surfaces by Liu (1985, 1986), Kaplan and Gray (1985), Liu *et al* (1986), Kaplan *et al* (1986, 1987) and Geertsma *et al* (1989); the use of Koch curves as sections of rough surfaces by Nyikos and Pajkossy (1985), Pajkossy and Nyikos (1986, 1988, 1989), Wang (1988) and de Levie (1989); the AC response of fractal networks by Clerc *et al* (1984, 1985, 1990) Yu *et al* (1986) and Dissado and Hill (1988); and the examination of porous Sierpinski carpet electrodes by Sapoval (1987) and Hill and Dissado (1988). A common feature of these analyses has been the development of constant phase angle (CPA) response in either the electrical impedance or admittance.

The CPA behaviour arises from the development of a frequency region over which the impedance (admittance) at angular frequency ω can be expressed (Liu 1986) by the scaling relationship

$$Z(\omega/\omega_0) = \chi Z(\eta\omega/\omega_0) \tag{1}$$

where ω_0 is the characteristic frequency of the system, χ the scaling parameter for the

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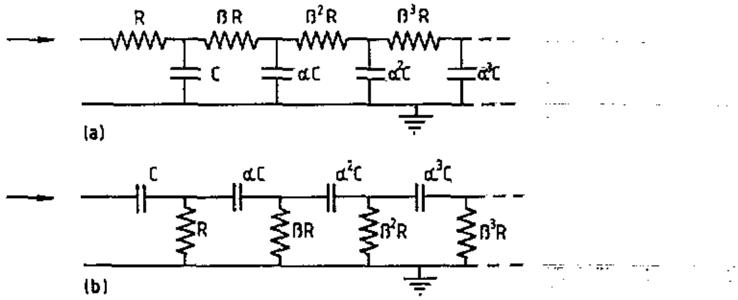

Figure 1. Two scaled transmission lines. In (*a*) the capacitors shunt the resistive line to earth whilst in (*b*) the elements have been exchanged so that the line is capacitive and coupled to earth through the resistances. In both cases the individual capacitors are scaled by α and the resistors by β. The lines can be either infinite or finite.

magnitude of the impedance Z and η the equivalent scaling parameter for the frequency ω. We define the CPA response from the relationship

$$Z(\omega/\omega_0) = Z_0(\mathrm{i}\omega/\omega_0)^{-\nu} \tag{2}$$

where substitution of (2) into (1) gives a solution of the latter when

$$\nu = \log\chi/\log\eta. \tag{3}$$

Furthermore, equation (2) can be expressed in the form

$$Z(\omega/\omega_0) = Z_0(\omega/\omega_0)^{-\nu}\,[\cos(\nu\pi/2) - \mathrm{i}\sin(\nu\pi/2)] \tag{4}$$

which shows that the phase angle $\nu\pi/2$ is constant and, in particular, independent of the frequency ω.

Here we consider a more general class of scaled circuit than those referred to earlier. This class is based on transmission lines containing resistance and capacitance components, multiplicatively scaled as shown in figure 1, to form scaled transmission lines (STL). We note that three of the circuits, those investigated by Liu (1985), Kaplan and Gray (1985) and Sapoval (1987), can be reconstructed into the form considered here, so that our conclusions are of general applicability.

When α and β, the scaling parameters of capacitance and resistance, respectively, are both unity, the scaled lines of figure 1 degenerate to the conventional ladder-like Cauer network and the properties of the line can be developed in terms of a propagation parameter (Connor 1972). When α and/or β are not unity, the propagation parameter becomes dependent on position in the line and no longer gives an effective description of the total line impedance (admittance). An exact technique for the analysis of scaled lines based on the continuous fraction approach has been developed (Liu 1985) and is used here to characterize the impedance properties, both in the infinite line case and for truncated lines of finite length.

In this analysis a second form of fractional power-law response (FPR) in the frequency domain has been obtained. FPR behaviour has already been reported for a scaled, Sierpinski-carpet, porous electrode immersed in an electrolyte (Hill and Disado 1988)

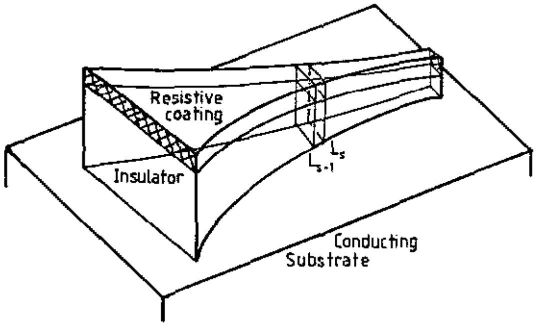

Figure 2. Schematic representation of a fractal transmission line. A resistive coating of constant thickness and width varying in a fractional power law form in length with exponent d_R lies on top of an insulating layer. The width of the insulator matches that of the coating but the thickness varies independently with length but in a fractional power law manner, with exponent d_C. Both layers are supported by a conducting substrate. The magnitude of the CPA exponent is d_R/d_C.

and in a deterministic scaled resistance/capacitance system (Dissado and Hill 1988). The FPR characteristic is given in terms of impedance by

$$Z(\omega/\omega_0) = Z_0 - Z_0(\mathrm{i}\omega/\omega_0)^{+\mu} \tag{5a}$$

$$Z(\omega/\omega_0) = Z_0\{1 - (\omega/\omega_0)^{\mu} [\cos(\mu\pi/2) + \mathrm{i}\sin(\mu\pi/2)]\} \tag{5b}$$

so that in terms of the real component of $Z(\omega/\omega_0)$ the fractional power law term forms a correction to Z_0, but for the imaginary component it is the *only* response, as it is for CPA. The constant, Z_0, is not an arbitrary number but the magnitude of the total response at zero frequency and hence an integral part of the complete description of the power law behaviour.

In the analysis developed here we use discrete resistance and capacitance elements to form the scaled line, as shown in figure 1. If we consider these elements as being infinitely small so as to form a continuum, and assume that the scaling is given by a cross-section dependence on the length along the line, as in figure 2, we can associate these dimensional properties with the resistance and capacitance, assuming constant resistivity and permittivity. Consider, for example, the sth element, at position L_s, with a length-dependent width; then the scaling parameter for the resistance may be expressed in the form

$$\ln[\beta] = d_R \ln[L_s/(L_s - 1)] \tag{6a}$$

where d_R is the dimension associated with the resistance in our continuum model. Assuming a suitable thickness variation in the insulating medium for the same model, we have

$$\ln[\alpha] = (d_C - d_R) \ln[L_s/(L_s - 1)] \tag{6b}$$

so that we may express our infinite-line scaling relationships in terms of the properties of an equivalent continuum model of a fractal transmission line.

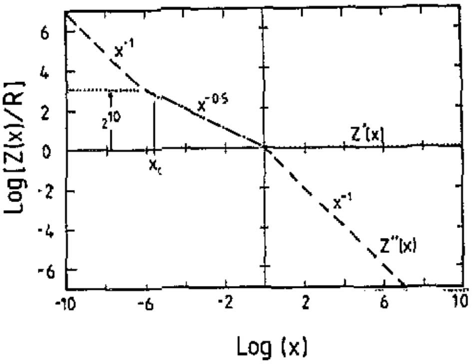

Figure 3. A log–log plot of the unit resistance normalized response as a function of frequency for the scaled transmission line shown in figure 1(*a*). $\alpha = \beta = 2$ and the number of scalings $N = 10$. From equation (10), when the scaling parameters are equal, the CPA behaviour has an exponent of 0.5 as shown in this diagram for $x < 1$. The truncation of the CPA behaviour at x_c is given from equation (11*b*) as a dispersion in the real component of Z of magnitude β^N (i.e. 2^{10}). For $x > 1$ the response is that of a resistance of magnitude R in series with a capacitance of magnitude C (i.e. the impedance of the $N = 0$ element).

2. Development of scaled transmission lines

2.1 *Simple lines,* CPA *behaviour*

The impedance of the scaled line shown in figure 1(*a*) (case (*a*)) can be written as the continuous fraction

$$Z(\omega) = R + \cfrac{1}{\mathrm{i}\omega C + \cfrac{1}{\beta R + \cfrac{1}{\mathrm{i}\alpha\omega C + \cfrac{1}{\beta^2 R + \cfrac{1}{\mathrm{i}\alpha^2\omega C + \cfrac{1}{\beta^3 R + \cfrac{1}{\mathrm{i}\alpha^3\omega C + \ldots}}}}}}} \qquad (7)$$

Dividing by R and substituting x for ωRC $(=\omega\tau)$ allows the continuous fraction to be expressed in the form

$$Z(x)/R = 1 + \beta/[\mathrm{i}\beta x + R/Z(\alpha\beta x)] \qquad (8)$$

This equation shows that the impedance of the infinite scaled transmission line can be represented by a recurrence relationship with the magnitude scaled multiplicatively by the factor $\beta(\equiv \chi)$ and the frequency, x, scaled separately by the product $\alpha\beta(\equiv \eta)$.

When the conditions

$$x < 1 \qquad (9a)$$

$$\beta Z(\alpha\beta x)/R < 1/\mathrm{i}x \qquad (9b)$$

$$\beta Z(\alpha\beta x)/R > 1 \qquad (9c)$$

apply, equation (8) can be written as the recurrence relation form of equation (1) which results in CPA behaviour, as shown in figure 3. The conditions (8) define an area in a

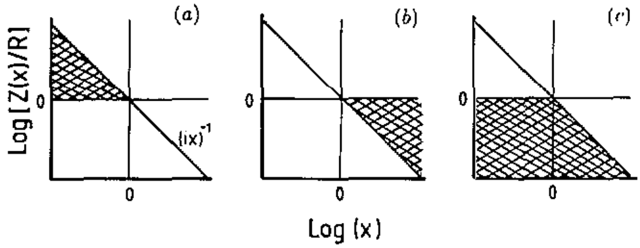

Figure 4. Schematic plots of the log of the response $Z(x)$ in terms of the elemental resistance R, as a function of the log of the frequency scaled by the elemental relaxation time RC. The shaded regions indicate the areas in which CPA or FPR behaviour is expected to be observed. The specific conditions (*a*) to (*c*) are considered in the text.

$\log(Z/R)$ versus $\log(x)$ plane, see figure 4(*a*), within which the CPA is observed. From equations (1) and (2), the magnitude of the exponent of the CPA response is

$$\nu = \log\beta/\log(\alpha\beta) \tag{10}$$

which for both $\beta, \alpha\beta > 1$ and $\beta, \alpha\beta < 1$ is positive, as required, giving a further condition on the observation of the CPA response. We note that for the fractal transmission line

$$\nu = d_R/d_C = d_w$$

which, following Lui (1986), defines the exponent ν as the fractal dimension of the CPA response of the impedance.

It is convenient when calculating the full response from equation (8) to limit the number of iterations. Machine computation can then be done by starting with the highest order term and working back to the first element. For the case shown in figure 3, the imaginary component of the impedance rises continuously at low frequencies ($\propto \omega^{-1}$, $x < x_c$) whilst the real component saturates. Approximating the limiting response by setting ω to zero we have, for the real component $Z'(\omega)$, that

$$Z(\omega)/R \simeq 1 + \beta + \beta^2 + \beta^3 + \dots \beta^N \tag{11}$$

so that

$$Z(x)/R \simeq \begin{cases} 1/(1-\beta) & \text{for } \beta < 1 \quad (11a) \\ \beta^N & \text{for } \beta > 1 \quad (11b) \end{cases}$$

where $N + 1$ is the number of elements in the line.

From equations (11*a*) and (11*b*) we can determine the frequency below which the impedance no longer follows CPA behaviour. Taking this frequency as x_c, the response in the CPA region is given by $A(\mathrm{i}x)^{-\nu}$ and, as our functions are normalized in terms of impedance by R and frequency by ω_0, the constant A is unity.

Hence

$$x_c = \begin{cases} [\beta^{-N}/\cos(\nu\pi/2)]^{1/\nu} & \beta > 1 \\ [(1-\beta)\cos(\nu\pi/2)]^{-1/\nu} & \beta < 1. \end{cases} \tag{12a}$$

Finally we note that condition (9*a*) requires that anomalous scaling behaviour is only observed for the scaled line of figure 1(*a*) for normalized frequencies less than unity

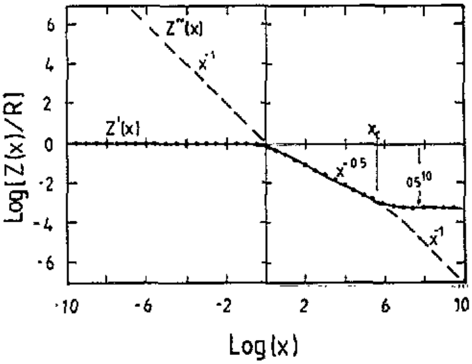

Figure 5. Response plot for the transmission line shown in figure 1(*b*). The inversion of the scaling parameters relative to those used to obtain figure 3 has resulted in an inversion of the characteristic shown in figure 3 through the point $x = 1$, $Z(x)/R = 1$, $\alpha = \beta = 0.5$, $N = 10$.

($\omega < 1/RC$) and when $\alpha > 1$ and $\beta > 1$. When $x > 1$ the impedance, from equation (8), can be approximated as

$$Z(x) = R(1 + 1/\mathrm{i}x) = R + 1/\mathrm{i}\omega C \tag{13}$$

which is the characteristic of a resistance of magnitude R in series with a capacitance of magnitude C; that is, the response of the first R/C element in the line.

Hence, for frequencies greater than the characteristic frequency of the first element, the current does not propagate along the line but is terminated by the (low) impedance of the $N = 0$ element. For frequencies small compared to x_c, the impedance characteristics of the line of finite length are essentially dominated by those of the final element in the line, see equation (11*b*), so that the complete line does not make a significant contribution to the total impedance. For a line for which N is infinite, x_c becomes zero and the scaled region extends to zero frequency. Therefore we have shown that the observation of CPA behaviour requires the bulk of the line to contribute to the impedance and the CPA exponent is a direct measure of the scaling nature of the elements forming the line.

A second STL example (case (*b*)) can be obtained by transposing the resistance and capacitance elements, as shown in figure 1(*b*). The development of the impedance relationship for this case follows an equivalent pattern, with the reduced scaling relation being obtained as

$$Z(x)/R = 1/\mathrm{i}x + \{1/[1 + R/\beta Z(\alpha\beta x)]\} \tag{14}$$

from which we can obtain the conditions for observation of CPA behaviour of equation (1) as

$$x > 1 \tag{15a}$$

$$\beta Z(\alpha\beta x)/R > 1/\mathrm{i}x \tag{15b}$$

$$\beta Z(\alpha\beta x)/R < 1. \tag{15c}$$

These conditions are shown in figure 4(*b*). The exponent of the CPA response, ν, is again given by equation (10) but with α and β less than unity. Examination of the infinite-frequency limit gives the limiting magnitudes of the impedance as

$$Z(x)/R \simeq \beta^N \qquad \beta < 1. \tag{16}$$

Consideration of these characteristics, and a comparison of figures 3 and 5, shows that this response is an inversion, through the datum point $Z(x)/R = 1$, $x = 1$, of that considered previously. Hence the theorem of inversion through a pole, which is a feature of conventional transmission lines, carries over into the scaled case.

2.2 Simple lines, FPR behaviour

Alternative solutions to equation (7) can be derived for a different range of parameters. When (case (*c*)) we have

$$\beta Z(\alpha\beta x)/R < 1/\mathrm{i}x \tag{17a}$$

and

$$1/[\mathrm{i}x + R/\beta Z(\alpha\beta x)] < 1 \tag{17b}$$

a solution of the form can be obtained

$$Z(x)/R = 1 + \beta Z(\alpha\beta x)/R. \tag{18}$$

Hill and Dissado (1988) have shown that a solution to equation (18) is the function

$$Z(x)/R = Z_0 - Z_0(\mathrm{i}x)^{\mu} \tag{19}$$

with

$$Z_0 = 1/(1-\beta) \tag{20a}$$

$$\mu = \ln\beta/\ln\alpha\beta \tag{20b}$$

which, for $\alpha > 1$ and $\beta < 1$ such that $\alpha\beta > 1$, defines the FPR characteristic with μ negative and with a modulus less than unity, $-1 < \mu < 0$. Equations (17) give the allowed

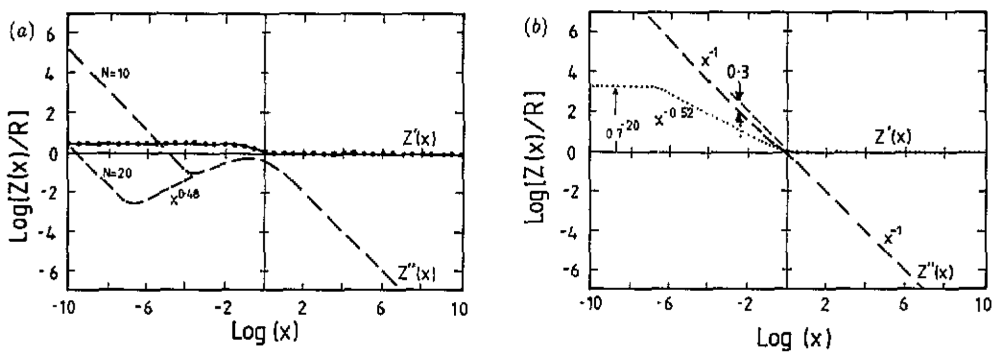

Figure 6. Two examples of FPR behaviour for the scaled transmission line of figure 1(*a*). In (*a*), $\alpha = 3.0$, $\beta = 0.7$, $N = 10$ and 20. Under these conditions, compare with figure 4(*c*), FPR behaviour develops in the quadrant $x < 1$ and $Z/R < 1$. For these particular values of scaling parameters the FPR exponent is +0.48. In this plot the effect of changing the value of N is shown; it is clear that as N approaches infinity the response plot takes the form of a single dispersion in Z' and a peak in the loss component. In (*b*), $\alpha = 0.7$, $\beta = 3.0$, $N = 20$. Exchange of the magnitudes of the parameters from those in (*a*) requires FPR behaviour to appear in the octant bounded by the unity response value and x^{-1}, as shown in figure 4(*a*). The magnitude of the FPR dispersion is α^N and $\mu = 1 + \ln\alpha/\ln\alpha\beta$.

area in the log–log plot of $Z(\omega)/R$ as a function of frequency, as in figure 4(*c*). Particular examples of FPR behaviour are presented in figure 6. In figure 6(*a*) the constant term for $x < 1$ is the resistance and a fractional power law of exponent +0.48 is found in the imaginary (i.e. capacitive) component and gives rise to a loss peak in the region $x \simeq 1$. In this figure we show the effect of varying the number of iterations. It is clear that, as N approaches infinity, the response will show a single dispersion in $Z'(x)$ and its accompanying loss peak.

Alternatively, we can assume (case (*d*)) that $Z(x)/R$ for $x < 1$ is greater than unity, so that from equation (8) we have that

$$Z(x)/R = \beta Z[\alpha\beta x]/[R + \mathrm{i}\beta x Z(\alpha\beta x)] \tag{21}$$

which has as a solution

$$Z(x)/R = A/\mathrm{i}x + D(\mathrm{i}x)^{-\mu} \tag{22}$$

with

$$A = 1 - \alpha \tag{23a}$$

$$\mu = 1 + \ln\alpha/\ln\alpha\beta \tag{23b}$$

for $\alpha < 1$ and $\beta > 1$ so that $\alpha\beta > 1$. The form of this response is shown in figure 6(*b*) with the FPR now appearing in the real component of the impedance and the capacitance showing a dispersion of magnitude 1-α.

In a similar manner (case (*e*)), considering equation (14) under the condition

$$\beta Z(\alpha\beta x)/R < 1 \tag{24}$$

we have

$$Z(x)/R = A/[\mathrm{i}x + \beta Z(\alpha\beta x)/R] \tag{25}$$

for which equation (22) is a solution (for $x < 1$) with A and μ given by

$$A = \alpha/(\alpha - 1) \tag{26a}$$

$$\mu = \ln\beta/\ln\alpha\beta \tag{26b}$$

with $\alpha > 1, \beta < 1$ and $\alpha\beta < 1$. A response of this form is shown in figure 7(*a*).

Alternatively, the term $(\mathrm{i}x)^{-1}$ can be neglected for $x > 1$ in equation (14), in which case (case (*f*)) the solution is given by

$$Z(x)/R = Z_0 + D(\mathrm{i}x)^{-\mu} \tag{27}$$

with

$$Z_0 = (\beta - 1)/\beta \tag{28a}$$

$$\mu = -\ln\beta/\ln\alpha\beta \tag{28b}$$

with $\alpha < 1, \beta >$ and $\alpha\beta < 1$. Note that μ is positive under these conditions.

Examples of these final sets of FPR characteristics are given in figure 7(*b*) and a summary of the properties of the fractional power responses is listed in table 1. It should be noted that the limitations on the ranges of the exponents ν and μ limit the ranges of

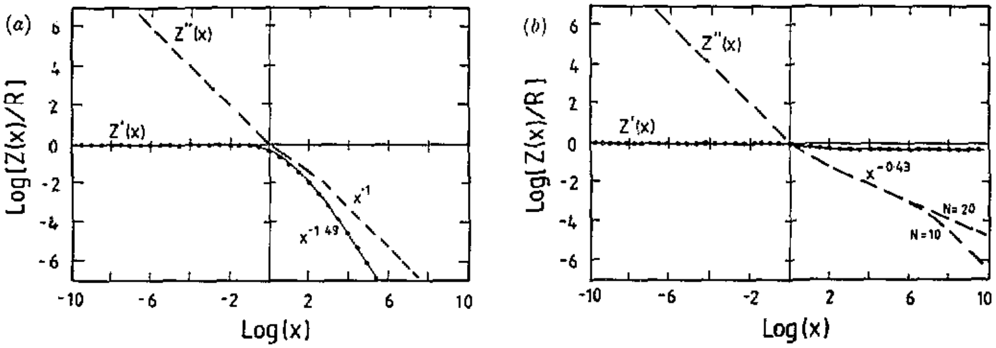

Figure 7. Two examples of FPR behaviour for the scaled transmission line of figure 1(*b*). In (*a*), $\alpha = 1.126$, $\beta = 0.7$, $N = 100$. This is the final form of FPR in which the dispersive component is the capacitance. Without scaling, the real component would be expected to follow the Debye characteristic for $\omega\tau > 1$ (i.e. $\propto \omega^{-2}$) but is now of a fractional power form with an exponent that lies between -1 and -2, and in this case is -1.49. In (*b*), $\alpha = 0.1$, $\beta = 2.0$. FPR behaviour of exponent -0.43 develops in the quadrant $x > 1$ and $Z/R < 1$. The response for $N = 10$ is truncated with x_r about 10^6.

Table 1. Summary of the power law regions of behaviour.

		α	β	$\alpha\beta$	Exponent range	Range of x	Model figure	Exponent function	Response figure
CPA	(*a*)	>1	>1	>1	$0 < \nu < 1$	<1	1(*a*)	$\ln\beta/\ln\alpha\beta$	3
	(*b*)	<1	<1	<1	$0 < \nu < 1$	>1	1(*b*)	$\ln\beta/\ln\alpha\beta$	5
FPR	(*c*)	>1	<1	>1	$-1 < \mu < 0$	<1	1(*a*)	$\ln\beta/\ln\alpha\beta$	6(*a*)
	(*d*)	<1	>1	>1	$0 < \mu < 1$	<1	1(*a*)	$1 + \ln\alpha/\ln\alpha\beta$	6(*b*)
	(*e*)	>1	<1	<1	$1 < \mu < 2$	>1	1(*b*)	$\ln\beta/\ln\alpha\beta$	7(*a*)
	(*f*)	<1	>1	<1	$0 < \mu < 1$	>1	1(*b*)	$-\ln\beta/\ln\alpha\beta$	7(*b*)

the scaling parameters α and β. The regions of behaviour for the six forms of response are shown on a log α/log β plot in figure 8; regions of CPA behaviour are shown singly shaded and those of FPR behaviour cross-shaded. Note that the limiting conditions are, in practice, α, β, $\alpha^2\beta$ and $\alpha\beta^2$ greater or less than unity.

As has already been indicated in figures 3, 5, 6 and 7, when no power law behaviour is present, the impedance characteristics for both examples degenerate to those of a resistance and capacitance in series connection, namely a constant resistance as the real component of the impedance and the inverse-frequency dependence of a constant capacitance as the loss component.

2.3 Complex lines

The $N = 0$ elements for four more complex STLs are shown in figure 9. In developing these elements, we retain the scaling variables α and β for capacitance and resistance with the further condition that *every* capacitance and resistance is required to be scaled.

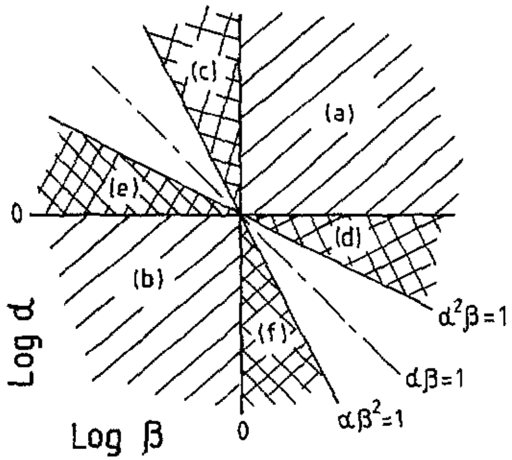

Figure 8. A schematic representation on a log α/log β plot of the regions of applicability of the model responses for simple scaled transmission lines from sections 2.1 and 2.2. The single shading quadrants are regions of CPA response and the cross-shaded areas those of FPR behaviour. The unshaded regions around $\alpha\beta = 1$ do not exhibit scaling behaviour. The labelling corresponds to that used in sections 2.1 and 2.2.

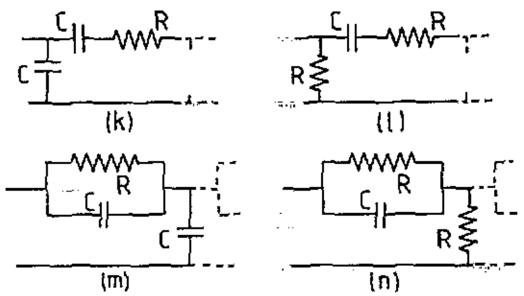

Figure 9. The four $N = 0$ circuit elements used as the basis for more complex fractal transmission lines. In the development of the lines all capacitances are scaled by α and all resistances by β at each iteration.

Using the continuous fraction development outlined in section 2.1, we have obtained for these STLs the conditions necessary for observation of CPA and FPR behaviour which are listed in table 2. In carrying through this work, it was found that the conditions for observation of fractional power law behaviour are more restrictive for these complex lines than for the simple cases described in sections 2.1 and 2.2. Indeed, the lines characterized by the elements (k) and (l) in figure 9 exhibit only simple parallel or series resistance/capacitance behaviour. The details of the approximations necessary for the development of scaled behaviour in the four circuits in figure 9 are given in table 2, in which it can be seen that the development of an anomalous frequency response in the circuits (k) and (l) requires conditions that are mutually exclusive. Furthermore, the only manner in which the STLs (m) and (n) may be manipulated to exhibit fractional power law behaviour was to develop them into the effective forms of figure 1 (i.e. as 'simple' scaled transmission lines). For both cases (m) and (n), CPA and FPR responses could then be obtained, as indicated in table 1.

Table 2. Summary of the reduced forms and conditions for the development of the circuits in figure 9.

Circuit	Reduced form	Conditions	Comments
9(k)	$\dfrac{Z(x)}{R} = \dfrac{1}{\mathrm{i}x + \dfrac{1}{1 + 1/\mathrm{i}x + \beta Z(\alpha\beta x)/R}}$	$\beta Z(\alpha\beta x)/R > 1/\mathrm{i}x$ and $<1/\mathrm{i}x$ for CPA $1/\mathrm{i}x > 1 + 1/\mathrm{i}x + \beta Z(\alpha\beta x)/R$ for FPR	Not developed Not developed
9(l) tt	$\dfrac{1}{1 + \dfrac{1}{1 + 1/\mathrm{i}x + \beta Z(\alpha\beta x)/R}}$	$\beta Z(\alpha\beta x)/R > 1$ and < 1 for CPA $1/\mathrm{i}x > 1 + 1\mathrm{i}x + \beta Z(\alpha\beta x)/R$ for FPR	Not developed Not developed
9(m)	$\dfrac{1}{1+\mathrm{i}x} + \dfrac{1}{\mathrm{i}x + R/\beta Z[\alpha\beta x]}$	$\beta Z(\alpha\beta x)/R > 1 > \mathrm{i}x$ for CPA $Z(x)/R = 1/(1+\mathrm{i}x) + \beta Z(\alpha\beta x)/R$ for FPR	Allowed Allowed
9(n)	$\dfrac{1}{1+\mathrm{i}x} + \dfrac{1}{1 + R/\beta Z[\alpha\beta x]}$	$\beta Z(\alpha\beta x)/R > 1/\mathrm{i}x < 1$ for CPA $Z(x)/R = 1 + \beta Z(\alpha\beta x)/R$ for FPR	Allowed Allowed

Typical response curves for cases (m) and (n) are shown in figures 10(*a*) and 10(*b*), respectively. For case (m), CPA behaviour is found for α and β greater than unity and FPR when α is greater than unity and β is less than unity. The equivalent responses have already been presented in figures 3 and 6(*a*), the essential difference being for x greater than unity, where the real impedance now decays as ω^{-2} instead of remaining constant, (i.e. the initial resistance and capacitance elements now act in parallel rather than in series). For case (n) the equivalent change with $Z''(\omega) \propto x^{+1}$, occurs at low frequencies ($x < 1$), and the CPA and FPR responses appear for x greater than unity, as shown in figure 10(*b*).

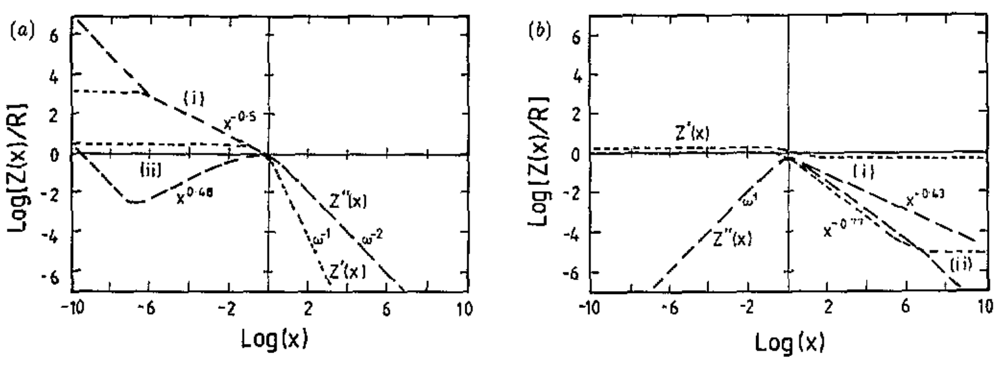

Figure 10. Two examples of scaled behaviour for the circuits in figure 9. In (*a*) are shown typical responses observed for case (m). (i) $\alpha = 2.0$, $\beta = 2.0$, $N = 10$ and (ii) $\alpha = 3.0$, $\beta = 0.7$, $N = 20$. Note particularly the changed response for $x > 1$ relative to that obtained for the equivalent simple line in figures 3 and 6(*a*). In (*b*) are shown typical responses for case (n) with (i) $\alpha = 0.1$, $\beta = 2.0$, $N = 20$ and (ii) $\alpha = 0.5$, $\beta = 0.1$, $N = 5$. Again the essential difference from figure 7(*b*) is not in the scaled region but in the nature of the unscaled response.

3. Discussion

In carrying out the development of the response of fractally scaled transmission lines we have chosen to work in terms of the frequency-dependent impedance $Z(\omega)$. Equivalent results can be derived for the admittance $Y(\omega)$ or the capacitance $C(\omega)$ where

$$Y(\omega) = 1/Z(\omega) \qquad C(\omega) = [\mathrm{i}\omega Z(\omega)]^{-1}. \tag{29}$$

In terms of CPA this means that the exponents for admittance have the same magnitude but are of opposite sign to those of impedance whereas the capacitance exponents are of the same sign but of magnitude $(1 - \nu)$, where ν is the impedance exponent (i.e. $C(\omega) \propto (\mathrm{i}\omega)^{\nu-1}$). Hence in all cases the CPA exponent is fractional. FPR behaviour is also a characteristic of all three methods of presentation, but the equivalences are not as direct and depend on the configuration of the individual elements, just as the developments of the circuits indicated in figure 8 are not all equivalent. As a basic rule, however, series connection in an impedance is equivalent to parallel connection in a capacitance and vice versa and it may be more useful to develop one or the other in a specific problem. Any one of the three descriptions is capable of being developed within the framework of the scheme used here.

We have deliberately kept to the α, β scaling notation so that the individual effects of the resistance and capacitance scalings can be followed. Since we have chosen to work in terms of unit resistance scaled impedance, $Z(x)/R$, β is identical to the magnitude scaling parameter χ of equation (1). This would not be the case if we had chosen to work in terms of capacitance, for which the magnitude scaling parameter would be α, although, in either case, the frequency scaling parameter η is given by $\alpha\beta$. The essential symmetry of the α, β scaling, which has already been shown in figures 3 and 5, is clearly shown in the parameter range plot of figure 8 in which the labels correspond to those used in sections 1 and 2 and in table 1. The single shaded quadrants are the regions of CPA behaviour and the four cross-shaded regions those in which FPR behaviour has been found. The apparent symmetry line $\alpha\beta = 1$ is the line along which there is no frequency scaling and for all magnitudes the elements add directly as resistors and capacitors in parallel or series. It should be noted however that whereas (*a*) and (*b*), and (*c*) and (*d*), exhibit symmetry about the $x = 1$, $Z/R = 1$, point, the high-frequency responses (*e*) and (*f*) do not. This follows from the basic requirement that the real component of the complex impedance has to be an increasing function with decreasing frequency.

In table 3 we present a summary of the scaled circuits reported in the literature which can be represented in the scaled transmission line form. It is clear from the table that the first three entries use only a single variable scaling factor, that of frequency, and although the networks have been constructed from different scaling models, this has been done in such a manner as to retain frequency scaling. The final case in table 3 is that proposed by Sapoval (1987), and can be seen to be equivalent to the Cantor block

Table 3. Summary of published models for fractally scaled metal/electrolyte interfaces. Here a is a length scaling parameter and N is the number of new pores generated per iteration.

Model	Author and figure	α	β	$\alpha\beta$
Rough Cantor bar	Liu (1985), figure 2	2	$a/2$	a
Rough surface, regular scaling	Kaplan and Gray (1985), figure 2	2	$a/2$	a
Self-affine Cantor block	Kaplan *et al* (1987). Figure 2	$4/a$	$a^2/4$	a
Porous Serpinski carpet electrode	Sapoval (1987) and Hill and Dissado (1988)	$(Na)^{-1}$	Na^2	a

scaling proposed by Kaplan *et al.* (1987) with the additional freedom of magnitude scaling by way of the multiplicative number of pores per iteration. It was for this model that the FPR response was first determined (Hill and Dissado 1988), a response that we have shown to be of general applicability as indicated in table 1, where the Sapoval case is case (*d*).

A novel feature of this work is that experimental observation of scaling responses, whether of CPA or FPR forms, has been shown to require that the sample on which the observation has been made can be represented, electrically, by the simple scaled transmission lines shown in figure 1. Although the nature of the elements forming the line may be complex, as indicated in figure 9, the macroscopically averaged response as given by CPA or FPR is insensitive to the microscopic detail and hence no conclusions can be made about the form of the material *on that scale*. However, if the observed response is truncated at both high and low frequencies, then we have access to the resistance and capacitance values of the unscaled elementary unit, R and C, and to the ranges βN and αN. This by itself is insufficient to characterize the scaling because the number of elements N cannot be determined. Only if a means of determining N, α or β is available can the transmission line be determined completely. One technique that is available from this work is to make use of the $\alpha, \beta < 1$ limits when these are available. As we have determined for these cases, the magnitudes of the relevant dispersions are simple functions of α or β, as for example equations (11*a*), (11*b*) and (20*a*). Once one of the scaling factors is known, the CPA exponent gives the other, N can then be determined, and the model characterized. In this sense FPR behaviour is more informative than CPA because, although the exponent of the power law component is a function of the scaling parameters, as for CPA, the dispersion in the non-power law component is only a function of one of the two scaling parameters, see equations (20*a*), (23*a*), (26*a*) and (28*a*). Hence the FPR systems are solvable, at least in terms of 'simple' scaled transmission lines.

Both CPA and FPR behaviour have been observed in the capacitive response of a wide range of physical and biological materials (Cole 1972, Jonscher 1983, Dissado 1987). In many cases not one, but two, constant phase regions have been found (Hill 1978, 1981) which, in terms of the approach used here, would imply the presence of two separately scaled relaxation mechanisms.

4. Conclusions

A number of scaled transmission lines have been examined and their electrical impedance properties are reported. Both CPA and FPR behaviour have been found and the conditions, in terms of scaling magnitudes and frequency ranges, for the observation of these response functions have been determined. it has also been shown that experimental observation of these responses allows detailed analysis of the scaling element and range of scaling, particularly with FPR behaviour.

The relationship between the scaled transmission lines and fractal lines has been examined and it has been shown that for particular cases the frequency exponent of the CPA and FPR dispersions can be considered as a fractional dimension for the impedance (or admittance) response of an equivalent fractal transmission line.

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